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Lot Size-Reorder Point Inventory Model with Fuzzy Demands

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Abstract—This paper discusses the lot size-reorder point inventory problem with fuzzy demands. Different from the existing studies, the shortages are backordered with shortage cost incurred. The α cut of the fuzzy demand is used to construct the fuzzy total inventory cost for each inventory policy (Q, r), where Q is the quantity to be ordered and r is the reorder point. Yager's ranking method for fuzzy numbers is utilized to find the best inventory policy in terms of the fuzzy total cost. Five pairs of simultaneous nonlinear equations for the optimal Q^* and r^* are derived for r in five different ranges of the fuzzy demand. When the demand is a trapezoidal fuzzy number, each pair of the simultaneous equations reduces to a set of closed-form equations. They are proved to be able to produce the optimal solution. Apparently, the methodology developed in this paper can be applied to other types of inventory problems to find the best inventory policy.

Keywords—Fuzzy sets, Inventory, Optimization.

INTRODUCTION

Inventories represent an important asset to a business operation. The proper control of their levels usually brings significant savings in costs. The development of inventory theory has evolved through several stages since its beginning in the 1920s [1,2]. At first, it had very simple models that used only a few parameters to describe the nature of deterministic demands. Later, these models were embellished to include more details, and probabilistic models were developed in the 1950s to capture the effects of stochastic demands. However, in the real world applications, there are cases that the probability distribution of the demand is not obtainable due to lack of historical data. The introduction of a new product is a typical example. What a decision maker faces in this case is a fuzzy environment [3]. Under this circumstance, the demands are more suitably described by linguistic terms such as approximately equal to certain amount subjectively estimated by the expert, and the fuzzy set theory [4] provides a possible solution approach.

In the 1990s, several scholars began to develop models for the inventory problems under fuzzy environment. The problems they dealt with were uncertain parameters, such as the market price, shortage cost, and warehouse capacity. For example, Park [5] and Ishii and Konno [6] discussed the case of fuzzy cost coefficients. Roy and Maiti [7] developed a fuzzy economic order quantity (EOQ) model with a constraint of fuzzy storage capacity. Chang and Yao [8] solved the economic reorder point with fuzzy backorders. There were some articles addressing fuzzy

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demands. Lee and Yao [9] and Chang [10] investigated the economic production quantity model and Lee and Yao [11] and Yao and Lee [12,13] studied the EOQ model with fuzzy demands. A common characteristic of these studies is that shortages are backordered without extra costs. Moreover, the solutions such as the quantities to be ordered are fuzzy numbers rather than crisp values, which may cause difficulty for the decision maker to follow. Recall that in probabilistic models, although the demands are stochastic, the ordering quantity solved from the model is still a constant instead of a random number.

The nature of the inventory problem consists of repeatedly placing and receiving orders of given sizes at set intervals. From this standpoint, an inventory policy answers questions of "How much to order?" and "When to order?" In this paper, we discuss the inventory problem with fuzzy demands where backorders are permitted, yet a shortage cost is incurred. The decision variables are the ordering quantity Q, i.e., how much to order, and the reorder point r, i.e., when to order. Since the demand is fuzzy, the cost associated with each inventory policy (Q, r) is fuzzy as well. The approach of this paper is to find the (Q, r) with the minimum cost determined from a ranking method for fuzzy numbers.

The rest of this paper is organized as follows. First, the total inventory cost of the lot size-reorder point inventory problem is constructed from the α cut of the demand. Then a ranking method for fuzzy numbers is applied to derive the optimal ordering quantity and reorder point. Finally, an inventory problem with trapezoidal fuzzy demand is discussed to illustrate the solution method developed in this paper.

TOTAL INVENTORY COST

Suppose the demand of a commodity is uncertain. Its rate $\tilde{\lambda}$ is a fuzzy number described by the following general membership function:

$$\mu_{\tilde{\lambda}}(\lambda) = \begin{cases} L(\lambda), & l \le \lambda \le m, \\ 1, & m \le \lambda \le n, \\ R(\lambda), & n \le \lambda \le u, \end{cases} \tag{1}$$

where $L(\lambda)$ and $R(\lambda)$ are the left-shape and right-shape functions, respectively. Let Q and r denote the quantity to be ordered and the reorder point, respectively. The lead time for the placed order to arrive is a constant k. The costs considered are the ordering cost a (for each order), the unit cost c (for each item), the holding cost h (per unit time for each item), and the shortage cost p (per unit time for each item). For a specific demand rate λ, there are two cases to consider in calculating the total cost. One is $r \ge k\lambda$ and the other is $r \le k\lambda$. Since the lead time demand is $k\lambda$, no storage will occur if the reorder point r is greater than $k\lambda$. Referring to Figure 1a, the holding cost in one cycle is $0.5h[(Q + r - k\lambda) + (r - k\lambda)](Q/\lambda)$. In one year, there are λ/Q cycles. Therefore, the total cost per year is

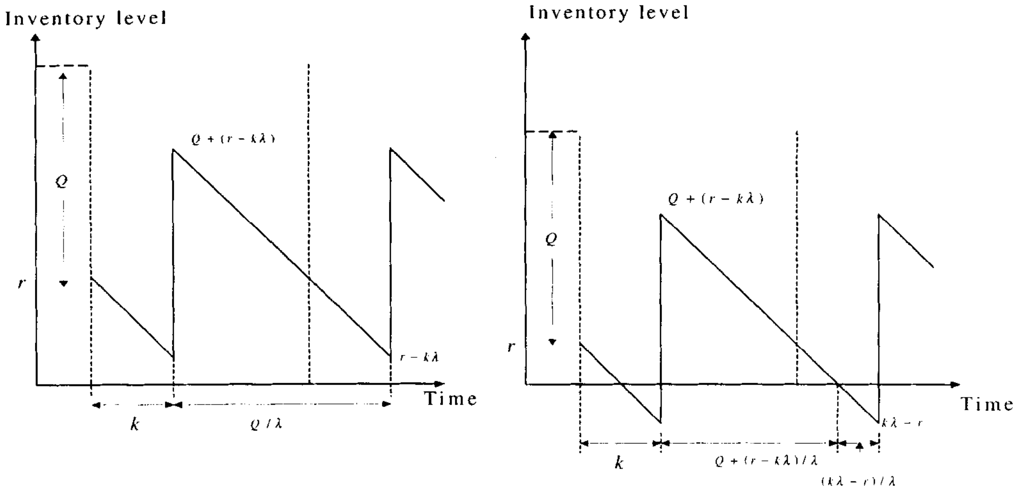
$$C^{+}(Q, r) = \frac{a\lambda}{Q} + c\lambda + h\left(\frac{Q}{2} + r - k\lambda\right). \tag{2}$$

On the other hand, if the reorder point r is smaller than $k\lambda$, then the inventory level reaches zero before the placement arrives. Referring to Figure 1b, the holding cost is $0.5h(Q + r - k\lambda)(Q + r - k\lambda)/\lambda$ and the shortage cost is $0.5p(k\lambda - r)(k\lambda - r)/\lambda$. In one year, the total cost is

$$C^{-}(Q, r) = \frac{a\lambda}{Q} + c\lambda + \frac{h}{2Q}\left(Q + r - k\lambda^2\right) + \frac{p}{2Q}\left(k\lambda - r\right)^2. \tag{3}$$

To summarize, the annual total cost for a specific demand rate λ is

$$C(Q, r \mid \lambda) = \begin{cases} C^{+}(Q, r), & r \ge k\lambda, \\ C^{-}(Q, r), & r \le k\lambda. \end{cases} \tag{4}$$



(a) $\lambda \le r/k$. (b) $\lambda \ge r/k$.

Figure 1. The inventory level for different demand rate λ.

In equation (1), different λ has different membership grade $\mu_{\tilde{\lambda}}(\lambda)$. Figure 2 shows a general shape for $\mu_{\tilde{\lambda}}$. Since $\tilde{\lambda}$ is fuzzy, the total cost $\tilde{C}(Q,r)$ is fuzzy as well. To construct the membership function for $\tilde{C}(Q,r)$ from the membership function of $\tilde{\lambda}$ directly is hardly possible. Another approach is to use the α cut. The α cut of $\tilde{\lambda}$ is

$$\left[L^{-1}(\alpha), R^{-1}(\alpha)\right] = \left[\min \mu_{\tilde{\lambda}}^{-1}(\alpha), \max \mu_{\tilde{\lambda}}^{-1}(\alpha)\right], \qquad 0 \le \alpha \le 1. \tag{5}$$

Different values of Q and r result in different $\tilde{C}(Q,r)$. There are five cases for the range of r to discuss, namely, $r/k \le l$, $l \le r/k \le m$, $m \le r/k \le n$, $n \le r/k \le u$, and $u \le r/k$.

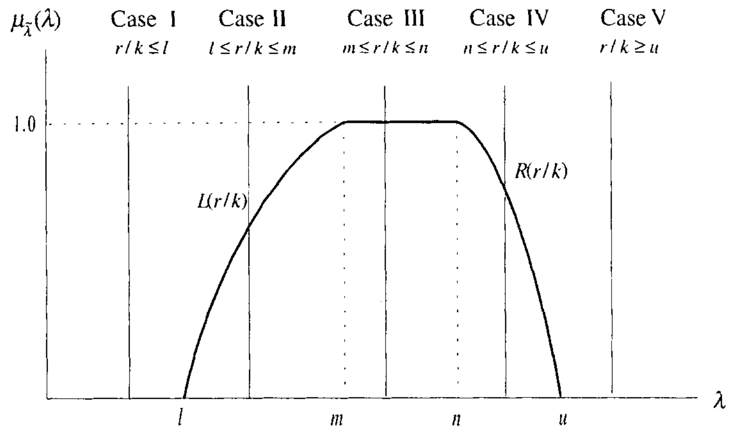


Figure 2. The membership function of $\tilde{\lambda}$.

CASE 1. $r/k \le l$. When $r/k \le l$, the demand rate λ is always greater than r/k since the support of $\tilde{\lambda}$ is $[l, u]$ (referring to Figure 2). Consequently, shortage will definitely occur. The associated cost is C^- defined in equation (3), and the α cut of $\tilde{C}(Q,r)$ is

$$C(\alpha) = \left[C^-\left(Q, r \mid \lambda = L^{-1}(\alpha)\right), C^-\left(Q, r \mid \lambda = R^{-1}(\alpha)\right)\right], \qquad 0 \le \alpha \le 1. \tag{6}$$

CASE 2. $l \le r/k \le m$. Since the condition of $\lambda \le r/k$ assures no shortage while $\lambda > r/k$ implies that shortage will occur, in this case, the total cost for $\lambda \le r/k$ corresponds to C^+ defined in equation (2) and the total cost for $\lambda > r/k$ corresponds to C^- defined in equation (3). Thus, referring to Figure 2, the α cut of $\tilde{C}(Q,r)$ is

$$C(\alpha) = \begin{cases} \left[C^+\left(q,r \mid \lambda = L^{-1}(\alpha)\right), C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], & 0 \le \alpha \le L\left(\frac{r}{k}\right), \\ \left[C^-\left(q,r \mid \lambda = L^{-1}(\alpha)\right), C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], & L\left(\frac{r}{k}\right) \le \alpha \le 1. \end{cases} \tag{7}$$

CASE 3. $m \le r/k \le n$. When r/k lies in the range of m and n, the total cost for $\lambda \le r/k$ corresponds to C^+ and corresponds to C^- for $\lambda > r/k$, no matter what value of α is. Therefore, the α cut of $\tilde{C}(Q,r)$ is

$$C(\alpha) = \left[C^+\left(Q,r \mid \lambda = L^{-1}(\alpha)\right), C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], \qquad 0 \le \alpha \le 1. \tag{8}$$

This is manifested from Figure 2.

CASE 4. $n \le r/k \le u$. This case is a symmetric counterpart of Case 2. In Figure 2, λ is always smaller than r/k for $\mu_{\tilde{\lambda}}(\lambda) = \alpha > R(r/k)$. However, for $\mu_{\tilde{\lambda}}(\lambda) = \alpha \le R(r/k)$, λ could be greater than or smaller than r/k. Therefore, we have

$$C(\alpha) = \begin{cases} \left[C^+\left(Q,r \mid \lambda = L^{-1}(\alpha)\right), C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], & 0 \le \alpha \le R\left(\frac{r}{k}\right), \\ \left[C^+\left(Q,r \mid \lambda = L^{-1}(\alpha)\right), C^+\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], & R\left(\frac{r}{k}\right) \le \alpha \le 1. \end{cases} \tag{9}$$

CASE 5. $u \le r/k$. This case is a symmetric counterpart of Case 1. When $u \le r/k$, the demand rate λ is always smaller than r/k, which implies that shortage will never occur. The corresponding cost is C^+ defined in equation (2), and the α cut of $\tilde{C}(Q,r)$ is

$$C(\alpha) = \left[C^+\left(Q,r \mid \lambda = L^{-1}(\alpha)\right), C^+\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right], \qquad 0 \le \alpha \le 1. \tag{10}$$

From the α cut of $\tilde{C}(Q,r)$, the membership function of $\tilde{C}, \mu_{\tilde{C}}$, can be constructed. Except for some simple cases, $\mu_{\tilde{C}}$ usually is too complicated to be derived analytically. Nevertheless, it can still be constructed numerically by enumerating different values of α.

In the next section, we shall derive the optimal ordering quantity Q^* and reorder point r^* in terms of the fuzzy total cost.

OPTIMAL SOLUTION

The total cost $\tilde{C}(Q,r)$ is a function of the ordering quantity Q and reorder point r. Different values of Q and r result in different $\tilde{C}(Q,r)$. The idea of this paper is to find the minimum $\tilde{C}(Q,r)$ via some ranking method for fuzzy numbers. In the literature, a lot of ranking methods have been proposed and discussed [14–18]. One popular method which lends itself to this purpose is Yager's method [14]. This method has an advantage of not requiring the knowledge of the explicit form of the membership functions of the fuzzy numbers to be ranked. Moreover, it is very simple to apply.

Yager's method [14] is to calculate a ranking index $I(\tilde{C})$ for the convex fuzzy number $\tilde{C}$ from its α cut $C(\alpha) = [C_\alpha^L, C_\alpha^U]$ according to the following formula:

$$I\left(\tilde{C}\right) = \int_0^1 \frac{1}{2}\left(C_\alpha^L + C_\alpha^U\right) d\alpha. \tag{11}$$

The optimal Q^* and r^* which produce the smallest ranking index $I(\tilde{C})$ can be discussed from five cases as categorized in the preceding section.

CASE 1. $r/k < l$. When r/k is smaller than l, the α cut of $\tilde{C}(Q,r)$ is described by equation (6). Yager's ranking index in this case is

$$I\left(\tilde{C}\right) = \frac{1}{2}\int_0^1 \left[C^-\left(Q,r \mid \lambda = L^{-1}(\alpha)\right) + C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha. \tag{12}$$

Its partial derivatives with respect to Q and r are

$$\begin{aligned} I'_Q\left(\tilde{C}\right) &= \frac{h}{2} - \frac{p+h}{4Q^2}\left\{\int_0^1 \left[r - kL^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[r - kR^{-1}(\alpha)\right]^2 d\alpha\right\} \\ &\quad - \frac{a}{2Q^2}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\ I'_r\left(\tilde{C}\right) &= h + \frac{p+h}{2Q}\left\{\int_0^1 \left[r - kL^{-1}(\alpha)\right] d\alpha + \int_0^1 \left[r - kR^{-1}(\alpha)\right] d\alpha\right\}. \end{aligned} \tag{13}$$

The necessary conditions for $I(\tilde{C})$ to attain the minimum are $I'_Q(\tilde{C}) = 0$ and $I'_r(\tilde{C}) = 0$, which can be rewritten as

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\int_0^1 \left\{\left[r - kL^{-1}(\alpha)\right]^2 + \left[r - kR^{-1}(\alpha)\right]^2\right\} d\alpha + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\ Q &= \frac{h+p}{h}\left\{\frac{k}{2}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha - r\right\}. \end{aligned} \tag{14}$$

By equating the square of the second equation with the first equation, r^* can be solved via some zero-finding algorithms [19,20]. The optimal Q^* is then solved by substituting r^* into the second equation.

A sufficient condition for the Q^* and r^* solved from equation (14) to attain the minimum is the convexity of $I(\tilde{C})$. For $I(\tilde{C})$ to be a convex function, its Hessian matrix must be positive definite [19], which requires $I''_Q(\tilde{C}) > 0$ and $I''_Q(\tilde{C})I''_r(\tilde{C}) - [I''_{Q,r}(\tilde{C})]^2 > 0$. Since

$$\begin{aligned} I''_Q\left(\tilde{C}\right) &= \frac{p+h}{2Q^3}\left\{\int_0^1 \left[r - kL^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[r - kR^{-1}(\alpha)\right]^2 d\alpha\right\} \\ &\quad + \frac{a}{Q^3}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha \end{aligned} \tag{15}$$

is always positive, the sufficient condition reduces to $I''_Q(\tilde{C})I''_r(\tilde{C}) - [I''_{Q,r}(\tilde{C})]^2 > 0$, which can be simplified to

$$\int_0^1 \left[L^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[R^{-1}(\alpha)\right]^2 d\alpha - \frac{1}{2}\left\{\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha\right\}^2 > 0. \tag{16}$$

CASE 2. $l \le r/k \le m$. When r/k lies under the left-shape function of $\tilde{\lambda}$, the α cut of $\tilde{C}(Q,r)$ is described by equation (7). Yager's ranking index in this case is

$$\begin{aligned} I\left(\tilde{C}\right) &= \frac{1}{2}\int_0^{L(r/k)} \left[C^+\left(Q,r \mid \lambda = L^{-1}(\alpha)\right) + C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha \\ &\quad + \frac{1}{2}\int_{L(r/k)}^1 \left[C^-\left(Q,r \mid \lambda = L^{-1}(\alpha)\right) + C^-\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha. \end{aligned} \tag{17}$$

The necessary conditions for a minimal $I(\tilde{C})$ are that the first partial derivatives $I'_Q(\tilde{C})$ and $I'_r(\tilde{C})$ vanish. This pair of conditions leads to

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\left\{\int_{L(r/k)}^{1}\left[r-kL^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[r-kR^{-1}(\alpha)\right]^2 d\alpha\right\} \\ &\quad + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha, \\ Q &= -\frac{h+p}{2h}\left\{\int_{L(r/k)}^{1}\left[r-kL^{-1}(\alpha)\right] d\alpha + \int_0^1 \left[r-kR^{-1}(\alpha)\right] d\alpha\right\}. \end{aligned} \tag{18}$$

The solution procedure of Case 1 can also be applied here to find the optimal Q^* and r^*.

Regarding the sufficient condition, since the second derivative

$$\begin{aligned} I''_Q\left(\tilde{C}\right) &= \frac{p+h}{2Q^3}\left\{\int_{L(r/k)}^{1}\left[r-kL^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[r-kR^{-1}(\alpha)\right]^2 d\alpha\right\} \\ &\quad + \frac{a}{Q^3}\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha \end{aligned} \tag{19}$$

is always positive, the Q^* and r^* solved from (18) attain the minimum if $I''_Q(\tilde{C}) \cdot I''_r(\tilde{C}) - [I''_{Q,r}(\tilde{C})]^2 > 0$. In other words, a sufficient condition is

$$\begin{aligned} &\frac{(p+h)^2}{4Q^2}\left[2-L\left(\frac{r}{k}\right)+\frac{r}{k}L'\left(\frac{r}{k}\right)\right]\left\{\int_{L(r/k)}^{1}\left[r-kL^{-1}(\alpha)\right]^2 d\alpha + \int_0^1 \left[r-kR^{-1}(\alpha)\right]^2 d\alpha\right\} \\ &\quad - \frac{(p+h)^2}{4Q^2}\left\{\int_{L(r/k)}^{1}\left[r-kL^{-1}(\alpha)\right] d\alpha + \int_0^1 \left[r-kR^{-1}(\alpha)\right] d\alpha\right\}^2 \\ &\quad + \frac{a(p+h)}{2Q^4}\left[2-L\left(\frac{r}{k}\right)+\frac{r}{k}L'\left(\frac{r}{k}\right)\right]\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha > 0. \end{aligned} \tag{20}$$

CASE 3. $m \le r/k \le n$. For r/k lying in the range of m and n, the α cut of $\tilde{C}(Q,r)$ is described by equation (8). Yager's ranking index becomes

$$I\left(\tilde{C}\right) = \frac{1}{2}\int_0^1 \left[C^{+}\left(Q,r \mid \lambda = L^{-1}(\alpha)\right) + C^{-}\left(Q,r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha. \tag{21}$$

Taking the partial derivatives with respect to Q and r and setting to zero yields the following necessary conditions for a minimum:

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\int_0^1 \left[r-kR^{-1}(\alpha)\right]^2 d\alpha + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha, \\ Q &= \frac{h+p}{2h}\left[k\int_0^1 R^{-1}(\alpha)\, d\alpha - r\right]. \end{aligned} \tag{22}$$

The solution procedure of Case 1 is then applied to find the optimal Q^* and r^*.

Similar to the previous two cases, the second derivative

$$I''_Q\left(\tilde{C}\right) = \frac{p+h}{2Q^3}\int_0^1 \left[r-kR^{-1}(\alpha)\right]^2 d\alpha + \frac{a}{Q^3}\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha \tag{23}$$

is always positive. Therefore, if $I''_Q(\tilde{C}) \cdot I''_r(\tilde{C}) - [I''_{Q,r}(\tilde{C})]^2 > 0$, which is equivalent to

$$\int_0^1 \left[R^{-1}(\alpha)\right]^2 d\alpha - \left[\int_0^1 R^{-1}(\alpha)\, d\alpha\right]^2 + \frac{2a}{k^2(p+h)}\int_0^1 \left[L^{-1}(\alpha)+R^{-1}(\alpha)\right] d\alpha > 0, \tag{24}$$

then the Q^* and r^* solved from (22) produce a minimal $I(\tilde{C})$.

CASE 4. $n \le r/k \le u$. When r/k lies under the right-shape function of $\tilde{\lambda}$, Yager's ranking index can be derived from equation (9) as

$$
\begin{aligned}
I\left(\tilde{C}\right) = &\frac{1}{2}\int_0^{R(r/k)} \left[C^+\left(Q, r \mid \lambda = L^{-1}(\alpha)\right) + C^-\left(Q, r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha \\
&+ \frac{1}{2}\int_{R(r/k)}^1 \left[C^+\left(Q, r \mid \lambda = L^{-1}(\alpha)\right) + C^+\left(Q, r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha.
\end{aligned} \tag{25}
$$

The necessary conditions for minimizing $I(\tilde{C})$ are derived by setting $I'_Q(\tilde{C})$ and $I'_r(\tilde{C})$ to zero, which are

$$
\begin{aligned}
Q^2 &= \frac{p+h}{2h}\int_0^{R(r/k)} \left[r - kR^{-1}(\alpha)\right]^2 d\alpha + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\
Q &= -\frac{h+p}{2h}\int_0^{R(r/k)} \left[r - kR^{-1}(\alpha)\right] d\alpha.
\end{aligned} \tag{26}
$$

The same solution procedure of Case 1 is applied here to find the optimal Q^* and r^*.

Again, the second derivative

$$
I''_Q\left(\tilde{C}\right) = \frac{p+h}{2Q^3}\int_0^{R(r/k)} \left[r - kR^{-1}(\alpha)\right]^2 d\alpha + \frac{a}{Q^3}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha \tag{27}
$$

is positive. The sufficient condition for the Q^* and r^* solved from (26) to attain minimum is $I''_Q(\tilde{C}) \cdot I''_r(\tilde{C}) - [I''_{Q,r}(\tilde{C})]^2 > 0$. That is,

$$
\begin{aligned}
&\frac{(P+h)^2}{4Q^2}\left\{\left[R\left(\frac{r}{k}\right) - \frac{r}{k}R'\left(\frac{r}{k}\right)\right]\int_0^{R(r/k)} \left[r - kR^{-1}(\alpha)\right]^2 d\alpha - \left[\int_0^{R(r/k)} \left(r - kR^{-1}(\alpha)\right) d\alpha\right]^2\right\} \\
&\qquad + \frac{a(p+h)}{2Q^4}\left[R\left(\frac{r}{k}\right) - \frac{r}{k}R'\left(\frac{r}{k}\right)\right]\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha > 0.
\end{aligned} \tag{28}
$$

CASE 5. $u \le r/k$. Based on equations (10) and (11), Yager's ranking index in this case is

$$
I\left(\tilde{C}\right) = \frac{1}{2}\int_0^1 \left[C^+\left(Q, r \mid \lambda = L^{-1}(\alpha)\right) + C^+\left(Q, r \mid \lambda = R^{-1}(\alpha)\right)\right] d\alpha. \tag{29}
$$

The first partial derivatives with respect to Q and r are

$$
I'_Q\left(\tilde{C}\right) = \frac{h}{2} - \frac{a}{2Q^2}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \qquad I'_r\left(\tilde{C}\right) = h. \tag{30}
$$

Since $I'_r(\tilde{C})$ is a positive constant, $I(\tilde{C})$ is an increasing function with respect to r. Therefore, r should be set to its lower bound ku to minimize $I(\tilde{C})$. Now,

$$
I''_Q\left(\tilde{C}\right) = \frac{a}{Q^3}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha \tag{31}
$$

is always positive, indicating that $I(\tilde{C})$ is convex with respect to Q for any value of r. Therefore, setting $I'_Q(\tilde{C})$ to zero and r to its lower bound produces the optimal Q^* and r^*. Restated,

$$
Q^2 = \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \qquad r = ku. \tag{32}
$$

Notably, in Case 4, Q^2 as defined in equation (26) boils down to the Q^2 defined in equation (32) when $r = ku$. This implies that Case 5 can be covered by Case 4. Therefore, it is not necessary to discuss Case 5.

For each of the above cases, there is a restriction on the range of r that the corresponding equations can be applied. Hence, if the r^* solved from the pair of necessary conditions does not lie in the proper range, then that solution should be discarded.

When $\mu_{\tilde{\lambda}}$ has simpler forms, the necessary conditions for Q and r to attain the minimum usually reduce to a pair of closed-form equations, which are fairly easy to solve. In the next section, we use the trapezoidal fuzzy number to illustrate this.

TRAPEZOIDAL FUZZY DEMAND

Consider a common case that the demand rate $\tilde{\lambda}$ is described by the trapezoidal fuzzy number

$$\mu_{\tilde{\lambda}}(\lambda) = \begin{cases} \dfrac{(\lambda - l)}{(m-l)}, & l \le \lambda \le m, \\ 1, & m \le \lambda \le n, \\ \dfrac{(u-\lambda)}{(u-n)}, & n \le \lambda \le u. \end{cases} \tag{33}$$

The simultaneous equations for solving Q and r and the sufficient conditions for the four cases are simplified as follows.

CASE 1. $r/k \le l$.

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\left[(r-ku)(r-kn) + \frac{k^2(u-n)^2}{3} + (r-kl)(r-km) + \frac{k^2(m-l)^2}{3}\right] \\ &\quad + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\ Q &= \frac{h+p}{h}\left[\frac{1}{4}k(l+m+n+u) - r\right]. \end{aligned} \tag{34}$$

The sufficient condition stated in equation (16) becomes

$$\frac{(l+m+n+u)^2}{4} + \frac{(m-l)^2 + (u-n)^2}{6} > 0, \tag{35}$$

which evidently holds.

CASE 2. $l \le r/k \le m$.

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\left[(r-ku)(r-kn) + \frac{k^2(u-n)^2}{3} + (r-kl)(r-km)\right. \\ &\quad \left. + \frac{k^2(m-l)^2}{3} - \frac{(r-kl)^3}{3k(m-l)}\right] + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\ Q &= \frac{h+p}{2h}\left[\frac{(r-kl)^2}{2k(m-l)} - 2r + \frac{1}{2}k(l+m+n+u)\right]. \end{aligned} \tag{36}$$

The sufficient condition stated in equation (20) is very difficult to check. Another way is to separate Yager's ranking index function of equation (17) into two parts. If both parts are convex functions, then the aggregated function is convex as well. Let the holding cost part of $0.5\int_0^{L(r/k)} C^+(Q, r \mid \lambda = L^{-1}(\alpha))\, d\alpha$, plus the holding cost and shortage cost parts of $0.5\int_{L(r/k)}^1 C^-(Q, r \mid \lambda = L^{-1}(\alpha))\, d\alpha$ be $V(\tilde{C})$ and the rest be $U(\tilde{C})$. Restated, we have

$I(\tilde{C}) = U(\tilde{C}) + V(\tilde{C})$. It is not difficult to derive the sufficient condition for $U(\tilde{C})$ to be convex as

$$\int_0^1 \left[R^{-1}(\alpha)\right]^2 d\alpha - \left[\int_0^1 R^{-1}(\alpha)\right]^2 > 0, \tag{37}$$

and the sufficient condition for $V(\tilde{C})$ to be convex as

$$\left[1 - L\left(\frac{r}{k}\right)\right] \int_{L(r/k)}^1 \left[L^{-1}(\alpha)\right]^2 d\alpha - \left[\int_{L(r/k)}^1 L^{-1}(\alpha)\, d\alpha\right]^2 > 0. \tag{38}$$

If both conditions (37) and (38) hold, then $I(\tilde{C})$ is convex.

When $\tilde{\lambda}$ is trapezoidal, conditions (37) and (38) become

$$\frac{(u-n)^2}{12} > 0, \qquad (m-l)^2 \left[1 - L\left(\frac{r}{k}\right)\right]^4 > 0, \tag{39}$$

which obviously hold.

CASE 3. $m \le r/k \le n$.

$$\begin{aligned} Q^2 &= \frac{p+h}{2h}\left[(r-ku)(r-kn) + \frac{k^2(u-n)^2}{3}\right] + \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha, \\ Q &= \frac{h+p}{2h}\left[\frac{k(u+n)}{2} - r\right]. \end{aligned} \tag{40}$$

The third term of the sufficient condition stated in equation (24) is always positive. Therefore, a sufficient condition is that the first two terms be positive. When $\tilde{\lambda}$ is trapezoidal, it becomes

$$\frac{(u-n)^2}{12} > 0, \tag{41}$$

which clearly holds.

CASE 4. $n \le r/k \le u$.

$$\begin{aligned} Q^2 &= \frac{a}{h}\int_0^1 \left[L^{-1}(\alpha) + R^{-1}(\alpha)\right] d\alpha - \frac{p+h}{2h} \cdot \frac{(r-ku)^3}{3k(u-n)}, \\ Q &= \frac{(h+p)(ku-r)^2}{4kh(u-n)}. \end{aligned} \tag{42}$$

In equation (28), $R(r/k) - (r/k)R'(r/k) \ge R(r/k) \ge 0$ because $R'(r/k)$ is negative. Therefore, the third term is positive. Consequently, the sufficient condition simplifies to

$$R\left(\frac{r}{k}\right) \int_0^{R(r/k)} \left[r - kR^{-1}(\alpha)\right]^2 d\alpha - \left[\int_0^{R(r/k)} \left(r - kR^{-1}(\alpha)\right) d\alpha\right]^2 > 0, \tag{43}$$

which is

$$k^2(u-n)^2 \left[R\left(\frac{r}{k}\right)\right]^4 > 0, \tag{44}$$

when $\tilde{\lambda}$ is trapezoidal. Clearly, this sufficient condition always holds.

In all four cases, the corresponding sufficient conditions are satisfied. Therefore, the solutions derived from the necessary conditions are optimal provided the range requirement for r is satisfied.

EXAMPLE. As an illustration, consider an inventory problem with a trapezoidal demand rate $\tilde{\lambda} = (l, m, n, u) = (4000, 7000, 9000, 12000)$. Suppose the parameters are $(c, a, h, p) = (20, 30, 3, 10)$. Substituting these values into equation (34), the optimal Q_I^* and r_I^* are solved as $(Q_I^*, r_I^*) =$

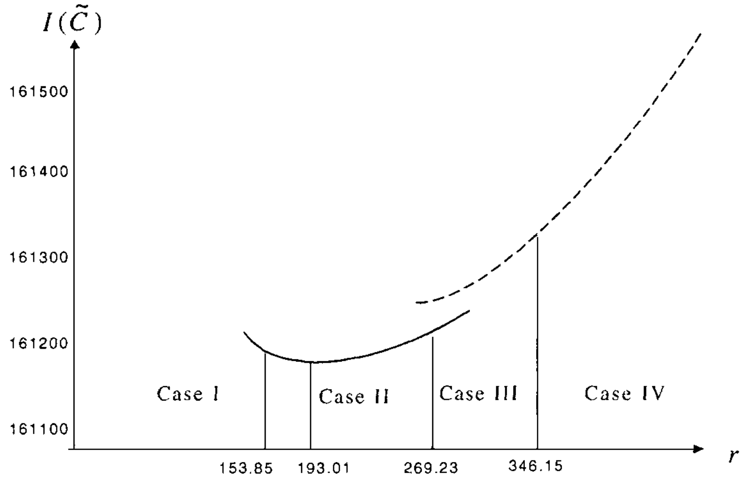
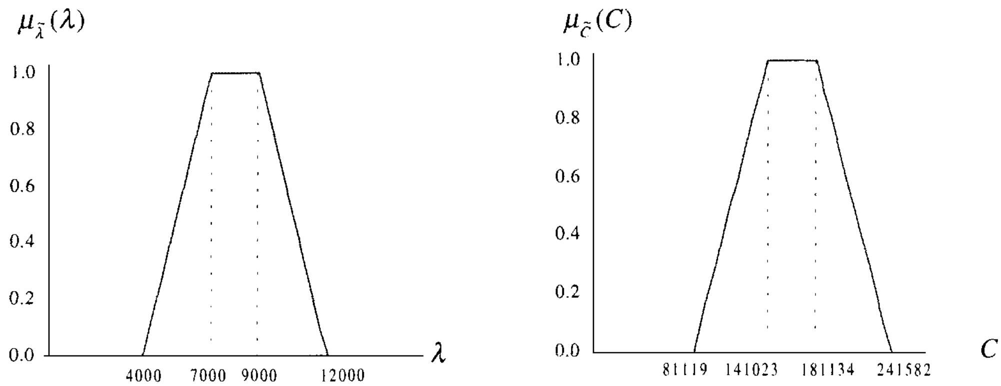


Figure 3. The $I(\tilde{C})$ curves for the four cases of the example.



(a) $\tilde{\lambda}$. (b) $\tilde{C}$.

Figure 4. The membership function of $\tilde{\lambda}$ and $\tilde{C}$ of the example.

$(516.07, 188.60)$ with $I_I(\tilde{C}) = 161190.94$. Since r_I^* does not lie in the required range of $[0, 153.85]$, this set of solutions is not valid.

Similarly, equations (36), (40), and (42) are applied for Cases 2, 3, and 4, respectively. The results are summarized in Table 1. Clearly, only the r_{II}^* solved from Case 2 satisfies the range requirement. Therefore, the optimal solution for this problem is to order $Q_{II}^* = 511.36$ items when the inventory level drops down to the reorder point $r_{II}^* = 193.01$. The corresponding ranking index is $I_{II}^*(\tilde{C}) = 161190.03$. Note, that although the ranking index of Case 3 is the smallest, its corresponding r_{III}^* fails to satisfy the range requirement. Figure 3 depicts the $I(\tilde{C})$ curves for the four cases.

Table 1. The solutions of the four cases for the example.

Case	Range of r	(Q^*, r^*)	$I(\tilde{C})$	Status
I	[0, 153.85]	(516.07, 188.60)	161190.94	not valid
II	[153.85, 269.23]	(511.36, 193.01)	161190.03	valid
III	[269.23, 346.15]	(549.17, 150.39)	161175.61	not valid
IV	[346.15, 461.54]	(482.67, 234.79)	161229.31	not valid

Figure 4a shows the membership function $\mu_{\tilde{\lambda}}$ of the fuzzy demand $\tilde{\lambda}$ of this example and Figure 4b is the membership function of the fuzzy total cost $\tilde{C}(Q^*, r^*)$ corresponding to the optimal solution. The membership function $\mu_{\tilde{C}}$ looks like trapezoidal, it is actually not. Another characteristic is that $\mu_{\tilde{C}}$ and $\mu_{\tilde{\lambda}}$ have similar shape, only the scale of the former is approximately forty times of the latter.

CONCLUSION

The inventory problems are widely studied in the literature. This paper discusses the case that shortages are backordered with shortage cost incurred. Different from the conventional studies is that the demands are fuzzy rather than deterministic or probabilistic. The introduction of the shortage cost causes the associated model to be very complicated to solve. The idea of this paper is to use the α cut of the fuzzy demand to derive the α cut of the fuzzy total cost corresponding to each inventory policy (Q, r). The minimum cost is determined by Yager's ranking method for fuzzy numbers. According to the general shape of the membership function of the fuzzy demand, the derivation of the optimal solution is discussed from five cases. The necessary and sufficient conditions for an optimal solution are derived.

When the demand is a trapezoidal fuzzy number, the sufficient condition for an optimal solution is satisfied. The optimal ordering quantity Q^* and reorder point r^* are calculated from four pairs of closed-form simultaneous equations. An example shows that the solution method developed in this paper is very easy to apply. The fuzzy total cost corresponding to the best inventory policy may not be derived analytically. However, it can still be constructed numerically by enumerating different α values from its α cut. The example shows that the shape of the membership functions of the fuzzy demand and fuzzy total cost look alike, only the scales are different.

This paper provides a methodology for constructing the fuzzy total inventory cost when the demands are fuzzy. Yager's ranking method is utilized to find the minimal total cost. While the discussion of this paper is concentrated on the case of backorders when shortage occurs, the methodology is applicable to the case of lost-sales and other more complicated cases.

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